# Game Theory

Lecture 10

## Computing the Domination-Based Concepts

- **“Dominance”**
- **“Iterated Elimination of Dominated Strategies”**

# Identifying dominated strategies

- Recall that one strategy dominates another when the first strategy is always at least as good as the second, regardless of the other players' actions.
- In this lecture, we discuss some computational tools for identifying dominated strategies, and consider the computational complexity of this process.
- **Recall: Iterated removal of <u>strictly</u> dominated strategies**
  - eliminates the same set of strategies regardless of the elimination order, and
  - all Nash equilibria of the original game will be contained in the remaining set.
  - Thus, this method can be used to narrow down the set of strategies to consider before attempting to identify a sample Nash equilibrium.
  - In the worst case, this procedure will have no effect—many games have no dominated strategies.
  - In practice, however, it can make a big difference to iteratively remove dominated strategies before attempting to compute an equilibrium.

# Identifying dominated strategies (Cont'd)

- **Recall: Iterated removal of <u>weakly</u> dominated strategies**
  - ❑ Elimination order does make a difference: the set of strategies that survive iterated removal can differ depending on the order in which dominated strategies are removed.
  - ❑ Removing weakly dominated strategies can eliminate some equilibria of the original game.

➢ There is still a computational benefit to this technique:

  - ❑ Since no new equilibria are ever created by this elimination (and since every game has at least one equilibrium), at least one of the original equilibria always survives.
  - ❑ This is enough if all we want to do is to identify a sample Nash equilibrium.
  - ❑ Furthermore, iterative removal of weakly dominated strategies can eliminate a larger set of strategies than iterative removal of strictly dominated strategies and so will often produce a smaller game.

# Domination by a pure strategy

- Checking whether a (not necessarily pure) strategy $s_i$ for player *i* is (strictly; weakly) dominated by any pure strategy for *i*.
- Let us consider the case of strict dominance.
  - To solve the problem we must check every pure strategy $a_i$ for player *i* and every **pure-strategy profile for the other players** to determine whether there exists some $a_i$ for which it is never weakly better for *i* to play $s_i$ instead of $a_i$. If so, $s_i$ is strictly dominated.

```
forall pure strategies a_i ∈ A_i for player i where a_i ≠ s_i do
    dom ← true
    forall pure-strategy profiles a_{-i} ∈ A_{-i} for the players other than i do
        if u_i(s_i, a_{-i}) ≥ u_i(a_i, a_{-i}) then
            dom ← false
            break
    if dom = true then
        return true
return false
```

# Domination by a pure strategy

```
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            dom ← false
            break
    if dom = true then
        return true
return false
```

- The case of weak dominance can be tested using essentially the same algorithm, except that we must test the condition $u_i(s_i,a_{-i}) > u_i(a_i,a_{-i})$. Also, we need to do a bit more book-keeping:
  - We must also set dom ← false if there is not at least one $a_{-i}$ for which $u_i(s_i, a_{-i}) < u_i(a_i, a_{-i})$.
- For both definitions of domination, the complexity of the procedure is O(|**A**|), linear in the size of the normal-form game.

# Domination by a mixed strategy

- Recall that sometimes a strategy is not dominated by any pure strategy, but is dominated by some mixed strategy.
- We cannot use a simple algorithm like before to test whether a given strategy $s_i$ is dominated by a mixed strategy because these strategies cannot be enumerated.
- However, it turns out that we can still answer the question in polynomial time by solving a linear program.
  - To this end, we will assume that player i's utilities are strictly positive.
    - This assumption is without loss of generality since if any player i's utilities were negative, we could add a constant to all payoffs without changing the game.

# Domination by a mixed strategy

- Each flavor of domination requires a somewhat different linear program.
  - First, let us consider **strict domination** by a mixed strategy. This would seem to have the following straightforward LP formulation (indeed, a mere feasibility program).

$$
\begin{aligned}
&\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\
&p_j \geq 0 && \forall j \in A_i \\
&\sum_{j \in A_i} p_j = 1
\end{aligned}
$$

- While the constraints do indeed describe strict domination by a mixed strategy, they do not constitute a linear program.
  - The problem is that the constraints in linear programs must be weak inequalities.

# Strict Domination by a mixed strategy

- Instead, we must use the LP that follows:

$$\begin{aligned} &\text{minimize} && \sum_{j \in A_i} p_j && \\ &\text{subject to} && \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) \quad \textbf{(*)} && \forall a_{-i} \in A_{-i} \\ & && p_j \geq 0 && \forall j \in A_i \end{aligned}$$

- This LP simulates the strict inequality of constraint through the objective function.
  - Because no constraints restrict the $p_j$'s from above, this LP will always be feasible.
  - However, in the optimal solution the $p_j$'s may not sum to 1; indeed, their sum can be greater than 1 or less than 1.
  - In the optimal solution, the $p_j$'s will be set so that their sum cannot be reduced any further without violating constraint (*).
  - Thus for at least some $a_{-i} \in A_{-i}$ we will have:

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) = u_i(s_i, a_{-i}).$$

# Strict Domination by a <u>mixed</u> strategy

$$\begin{aligned}
\text{minimize} \quad & \sum_{j \in A_i} p_j \\
\text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) \quad \textbf{(*)} \quad \forall a_{-i} \in A_{-i} \\
& p_j \geq 0 \qquad \forall j \in A_i
\end{aligned}$$

- A strictly dominating mixed strategy therefore exists if and only if the optimal solution to the LP has objective function value strictly less than 1.
  - In this case, we can add a positive amount to each $p_j$ in order to cause constraint (*) to hold in its strict version everywhere while achieving the condition $\sum_j p_j = 1$.

# Weak Domination by a **mixed** strategy

- Again our inability to write a strict inequality will make things more complicated. However, we can derive an LP by adding an objective function to the feasibility program.

$$\begin{aligned}
\text{maximize} \quad & \sum_{a_{-i} \in A_{-i}} \left[ \left( \sum_{j \in A_i} p_j \cdot u_i(a_j, a_{-i}) \right) - u_i(s_i, a_{-i}) \right] \\
\text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) \quad \textbf{(*)} \qquad \forall a_{-i} \in A_{-i} \\
& p_j \geq 0 \qquad \forall j \in A_i \\
& \sum_{j \in A_i} p_j = 1
\end{aligned}$$

- Because of constraint (*), any feasible solution will have a nonnegative objective value.
- If the optimal solution has a strictly positive objective, the mixed strategy given by the $p_j$'s achieves strictly positive expected utility for at least one $a_{-i} \in A_{-i}$, meaning that $s_i$ is weakly dominated by this mixed strategy.

# Iterated dominance

- Finally, we consider the iterated removal of dominated strategies.
- We only consider pure strategies as candidates for removal;
  - indeed, as it turns out, it never helps to remove dominated mixed strategies when performing iterated removal.
  - It is important, however, that we consider the possibility that pure strategies may be dominated by mixed strategies.

# Iterated dominance (Cont'd)

- For both flavors of domination, it requires only polynomial time to iteratively remove dominated strategies until the game has been maximally reduced (i.e., no strategy is dominated for any player).

- A single step of this process consists of checking whether every pure strategy of every player is dominated by any other mixed strategy, which requires us to solve ***at worst***

  $\sum_{i \in N} |A_i|$ linear programs.

- Each step removes one pure strategy for one player, so there can be ***at most*** $\sum_{i \in N} (|A_i| - 1)$ steps.